1 Joints Problem

Definition 1.1. Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$. A joint of $\mathcal{L}$ is any point that lies in the intersection of three non-co-planar lines of $\mathcal{L}$. The Joints problem is to find suitable upper bounds to the maximum number of joints possible in L lines, for any given $L \geq 3$.

Every joint is a triple intersection point (ie intersection of three lines). Since, any two distinct lines intersect in at most 1 point, there are no more than $\binom{L}{2} = L^2$ triple intersection points in any set of L lines. If the lines intersecting at any such triple intersection point are non-co-planar, then this point is a joint. Hence, a crude upper bound for the joints problem is L^2.

Example 1. Consider an $S \times S \times S$ grid of points with $S \in \mathbb{N}$. Let $\mathcal{L}_G$ be the set of all axis-parallel lines that intersect the grid. Then, any three lines intersecting at a grid point in $\mathcal{L}_G$ are non-co-planar, and hence every one of the S^3 grid points is a joint. One can show that there are $L_G := 3S^2$ lines in $\mathcal{L}_G$ and $S^3 \leq 3\sqrt{3}S^3 = L_G^{3/2}$ joints.

Example 2. Let $\mathcal{L}_T$ be the 6 lines containing each of the $L_T := 6$ edges of a tetrahedron. Then, each vertex of the tetrahedron is a joint as the three edges intersecting at any vertex are non-co-planar. Hence there are $4 \leq 6\sqrt{6} = L_T^{3/2}$ joints in $\mathcal{L}_T$. Actually, one can show that any set of 6 lines has at most 4 joints (Exercise).

Example 3. We generalize the previous example from $S = 4$ planes to $S \geq 3$ planes. Suppose $\mathcal{L}_P$ is the set of lines in the intersection of any two planes in some given set of $S \geq 3$ planes in general position (ie any two planes of this set intersect in a line and any three planes of this set intersect in a point). Since every line in $\mathcal{L}_P$ is formed by intersection of 2 planes out of S planes, there are $L_P := \binom{S}{2}$ lines in $\mathcal{L}_P$. One can show that every triple intersection point is the unique point in the intersection of some three planes of the given S planes, and there are $\binom{S}{3}$ of those. Hence, the number of joints $\leq \binom{S}{3} = \frac{S(S-1)(S-2)}{6} \leq \left(\frac{S(S-1)}{2}\right)^{3/2} = L_P^{3/2}$.

2 Joints Problem With Polynomials

Lemma 2.1. Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ with J joints, and let $F : \mathbb{R}^3 \to \mathbb{R}$ be a some smooth function that vanishes on each $l \in \mathcal{L}$. Then, $\mathbf{p} \in \mathbb{R}^3$ is a joint of $\mathcal{L} \implies \nabla F(\mathbf{p}) = 0$.

Proof. We consider tangent vectors at the joint $\mathbf{p}$ to the three non-co-planar lines of $\mathbf{p}$ and show that the directional derivative of F wrt each of these at $\mathbf{p}$ is 0. Finally, we argue that since these three tangent vectors span $\mathbb{R}^3$, the total derivative ie gradient of F is 0 at $\mathbf{p}$, as well. □

Lemma 2.2. (Main Lemma) Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$ that has $J \neq 0$ joints. Then, there is a line in $\mathcal{L}$ that contains at most $3J^{1/3}$ joints.

Proof. Taking S to be the set of J joints of $\mathcal{L}$, we use the **parameter counting argument** to get a non-zero polynomial of degree at most $3J^{1/3}$ that vanishes on S. We take $f_{\min}$ to be the lowest degree polynomial among such polynomials. Arguing by contradiction, if each line in $\mathcal{L}$ contains $> 3J^{1/3}$ joints, then by the **vanishing lemma**, $f_{\min}$ vanishes on each line of $\mathcal{L}$ as its degree is atmost $3L^{1/3}$. Hence, by the previous lemma, the gradient of $f_{\min}$ is zero as well, and in particular any partial derivative of $f_{\min}$ is a polynomial of lower degree than $f_{\min}$ that vanishes on S, a contradiction. $\square$

Theorem 2.3. Any set $\mathcal{L}$ of L lines in $\mathbb{R}^3$ has at most $10L^{3/2}$ joints.

Proof. The strategy is to use the main lemma to remove lines (each containing at most $3J^{1/3}$ joints of $\mathcal{L}$) one at a time until there are no more lines (and hence no more joints) left, as follows,

$$J(\mathcal{L}) \leq J(\mathcal{L} \setminus \{l_1\}) + 3J^{1/3} \leq J(\mathcal{L} \setminus \{l_1, l_2\}) + 3(2)J^{1/3} \leq \ldots J(\mathcal{L} \setminus \{l_1, \ldots, l_L\}) + 3(L)J(\mathcal{L})^{1/3} = 3LJ(\mathcal{L})^{1/3}.$$

Hence, $J(\mathcal{L}) \leq 3^{3/2}L^{3/2} = 3\sqrt{3}L^{3/2} \leq 10.L^{3/2}$. $\square$

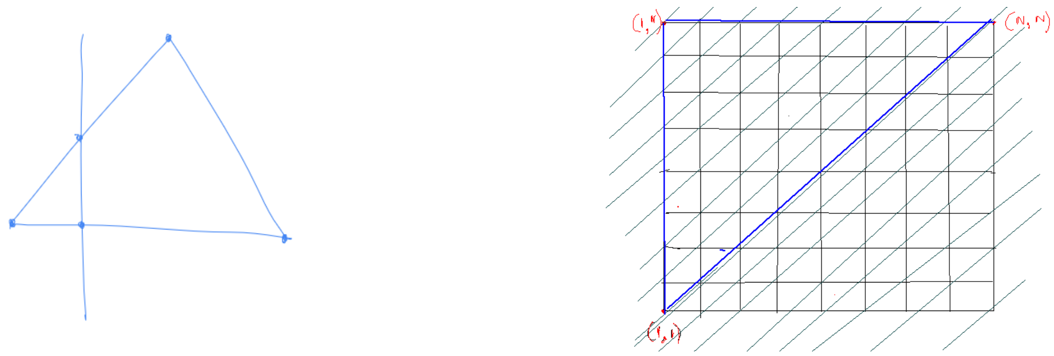
3 Joints Problem Without Polynomials

Instead of counting joints, since we are interested in the maximum number of joints of L lines, we could move these lines around in $\mathbb{R}^3$ hoping to maximize the number of triple intersection points that are joints, provided the total number of triple intersection points remains constant.

Definition 3.1. Suppose $\mathcal{L} = \{l_1, l_2, \ldots\}$ is a set of lines and $E = \{p_1, p_2, \ldots\}$ is a set of points. The incidence matrix is st the i, j-th entry is 1 if $p_j \in l_i$ and 0 otherwise. A perturbation of $(\mathcal{L}, E)$ is a pair $(\mathcal{L}', E')$, for some $\mathcal{L}'$ set of lines and E' set of points, st $I(\mathcal{L}, E) = I(\mathcal{L}', E')$.

So, we can try to rule out the possibility of perturbations with more than $CL^{3/2}$ joints. One way to do this is note that every triangle lies in a unique plane and can be uniquely characterized as three lines and three points such that every line has exactly two of these three points. Moreover, triangles are preserved by perturbations. Finally, using the following proposition, we can force more lines to be in the same plane as the three lines of a trinagle.

Proposition 3.1. Suppose (l_1, l_2, l_3) are the three lines of a triangle lying in a plane π. Let l be a line distinct from these three lines. Then, l intersects any two lines of the triangle $\implies$ l also lies in the plane π. (See Figure 1a for proof).



(a) Proof of Proposition 3.1 (b) Triangle Method for $(\mathcal{L}_0, E_0)$

Since, perturbations have same incidence matrices, these four lines are non-co-planar in any perturbation. This method, called the triangle method, shows (see Figure 1b) that at most 2 triple intersection points of $(\mathcal{L}_0, E_0)$ (where $\mathcal{L}_0$ is set of lines $x = c$, $y = c$, $y = x$ or $y = x \pm c$ for $c \in [1, \ldots, N]$; E_0 is set of triple intersection points of these lines) are joints under any perturbation, hence proving the joints problem statement for this particular configuration of lines. However, more generally, paper [1] is able to only show that L lines determine $o(L^2)$ joints, which is no more than an ϵ improvement from L^2. This is a consequence of existence of large triangle-free subsets.

Theorem 3.1. (Proposition 3.2 in [1]) For any $\epsilon > 0$, and for any large L, there is a set $\mathcal{L}$ of L lines in $\mathbb{R}^2$ and a set $F \subseteq \mathbb{R}^2$ st the number of triple intersection points in F is $\geq L^{2-\epsilon}$ and $(\mathcal{L}, F)$ is triangle free (ie has no triangle of $(\mathcal{L}, E)$).

While we do not prove this theorem in such a generality, we shall prove it for $(\mathcal{L}_0, E_0)$ (which, as we shall see, is already rather tedious to show without polynomials). Let $B \subseteq [1, \ldots, 2N]$. Define $F(B) := \{(x, y) \in [1, \ldots, N]^2 \, : \, x + y \in B\} \subseteq E_0$.

Lemma 3.2. $B \subseteq [1, \ldots, 2N]$ contains no 3-term progressions $\implies$ $(\mathcal{L}_0, F(B))$ has no triangles of $(\mathcal{L}_0, E_0)$.

Proof. Consider the tiangle in Figure 2. We argue that c_1, c_2, c_3 are a 3-term arithmetic progression in B by noting that $a_3 = a_2$, $b_2 = b_1$, $b_3 - a_3 = b_1 - a_1$, and hence, $c_3 - c_2 = a_3 + b_3 - a_2 - b_2 = b_3 - b_2 = b_3 - b_1 = a_3 - a_1 = a_2 - a_1 = a_2 + b_2 - a_1 - b_1 = c_2 - c_1$. □

Theorem 3.3. *(Behrend)* For any $\epsilon > 0$, for all N sufficiently large, there is $B \subseteq [-N, \ldots, N]$ (one just shifts each $i \in [-N, \ldots, N]$ by N) with no 3-term arithmetic progressions and $|B| >\sim N^{1-\epsilon}$.

To prove this theorem, we make use of the following lemma, called Behrends' construction.

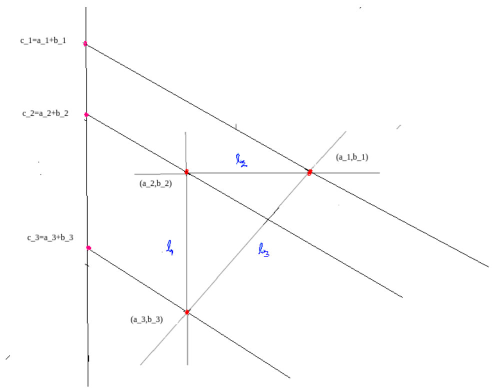


Figure 2: Correspondence between triangles and 3-term arithmetic progressions

Lemma 3.4. Let $n \in \mathbb{N}$ any. For any $S \geq 1$, there exists $A \subseteq [-S, \ldots, S]^n$ with no 3-term arithmetic progressions and $|A| \geq c(n)S^{n-2}$.

Proof. The integer lattice $[-S, \ldots, S]^n := \{\mathbf{x} \in \mathbb{Z}^n \ : \ x_i \in [-S, S]\}$ has $(2S+1)^n > S^n$ points. For each $\mathbf{x} \in [-S, \ldots, S]^n \subseteq \mathbb{Z}^n$, $|\mathbf{x}|^2 \in \{1, \ldots, nS^2\}$, as such, $[-S, \ldots, S]^n = \bigsqcup_{k=1}^{nS^2} A_k$ where $A_k := [-S, \ldots, S]^n \cap C_{n,k}$ for $k \in \{1, \ldots, nS^2\}$ and $C_{n,k}$, the n-sphere of radius $\sqrt{k}$. So, there are S^n objects to be filled in nS^2 places. Hence, by the pigeonhole principle, there is some A_M st $|A_M| \geq \frac{S^n}{nS^2} = \frac{1}{n}S^{n-2}$. We take $A := A_M$. Since points in A are on a sphere, any 3-term arithmetic progression of A must lie on a line, and any line intersects the sphere in at most 2 points, we are done. $\square$

Finally, we have the following result on existence of large traingle-free subsets of $(\mathcal{L}_0, E_0)$.

Theorem 3.5. For any $\epsilon > 0$, for all L sufficiently large, there is a subset $F \subseteq E_0$ st $(\mathcal{L}_0, F)$ is triangle free and $|F| \geq L^{2-\epsilon}$.

Proof. This follows from Behrend's theorem and Lemma 3.2. $\square$

4 Why Polynomial Methods Work?

There are $\sim D^n$ polynomials of degree at most D in n-variables. This gives us a lot of flexibility (D^n degrees of freedom, apriori). Using this, the parameter counting argument ensures that given a finite set S there is at least one non-zero polynomial of degree at most $n|S|^{1/n}$ that vanishes on the set S. On the other hand, the vanishing lemma says that polynomials of degree atmost D in n-variables that vanish at more than D points of a line, are identically zero on this line. That is polynomials behave rigidly on lines (with only D degrees of freedom).

Combining these two results, one gets that there is a non-zero polynomial that vanishes on a given set S, the degree of this polynomial is bounded by $n|S|^{1/n}$ and it cannot vanish at more than $n|S|^{1/n}$ points on a line. It is this gap, between D^n degrees of freedom and D degrees of freedom that allows us to use polynomial methods.

References

[1] L. Guth and A. Suk, The joints problem for matroids. J. Combin. Theory Ser. A 131 (2015), 71-87